5.4 Discrete random variables

Candidates should be able to:	Notes and examples
• draw up a probability distribution table relating to a given situation involving a discrete random variable X, and calculate $E(X)$ and $Var(X)$	
• use formulae for probabilities for the binomial and geometric distributions, and recognise practical situations where these distributions are suitable models	Including the notations $B(n, p)$ and $Geo(p)$. $Geo(p)$ denotes the distribution in which $p_r = p(1-p)^{r-1}$ for $r = 1, 2, 3, \ldots$.
• use formulae for the expectation and variance of the binomial distribution and for the expectation of the geometric distribution.	Proofs of formulae are not required.

GEOMETRIC DISTRIBUTION

BINOMIAL

Coin:

$n = 20$

$p = \text{Head} =$

no. of times success occur.

$P(X=8) = {}^{20}C_8 (0.5)^8 (0.5)^{12}$

Binomial looks at a repeated experiment where n = total repeats and r = no. of time success occur.

Geometric

1) Repeated
2) Success/failure
3) Discrete outcomes.

Repeat an experiment till success happens for the first time.

There is no n = no. of repeats involved.

Q: Roll a dice till it lands on 6.

success = $P(6) = \frac{1}{6}$ $\qquad p = \frac{1}{6}, \; q = \frac{5}{6}$

(i) Find probability that 6 lands on 8^{th} time we rolled the dice.

$$\begin{array}{ccccccccccc} F & F & F & F & F & F & F & S & & F^7 S \\ \end{array}$$

$$P(X=8) = \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)$$

$$P(X=8) = q^{8-1} p^1$$

$$\boxed{\text{GEOMETRIC DISTRIBUTION}}$$

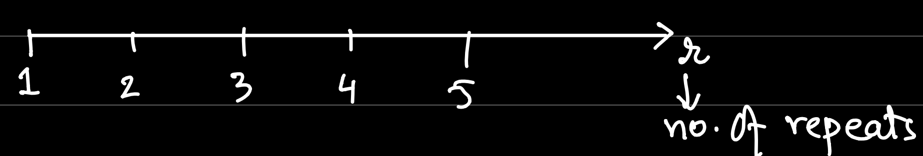
Repeat experiment until success occurs.

$$P(X = \underset{\downarrow}{r}) = q^{r-1} \cdot p$$

no of repeat where we want success to happen

$$P(X=6) = q^5 p$$
$$P(X=4) = q^3 p$$

IMPORTANT:



LETS CRAM

GEOMETRIC Repeat experiment until first success.

1. $P(X = r) = q^{r-1} \cdot p = (1-p)^{r-1} \cdot p$

   r ↓ no. of repeat on which we want first success.

2. Mean = Expected Value $= \frac{1}{p}$

3. $P(X \leq r) = 1 - q^r$

4. $P(X > r) = q^r$

Be really careful about $\leq$ and $>$ sign here.

Q Roll a dice till it lands on 6.

$p = \frac{1}{6}, \; q = \frac{5}{6}$

(i) Find probability that dice lands on 6 on fourth turn.

$$P(X=4) = q^{r-1} p = q^{4-1} p$$
$$= \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)$$

(ii) $P(X \leq 6) = 1 - q^r$

$$= 1 - \left(\frac{5}{6}\right)^6$$

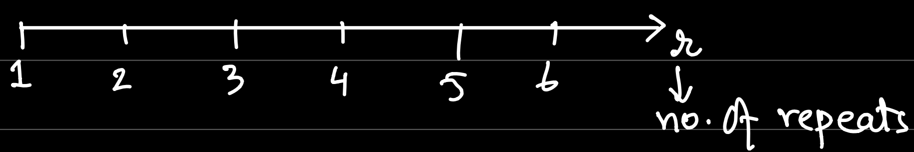
(iii) $P(X<6) = P(X \leq 5) = 1 - q^{r} = 1 - \left(\frac{5}{6}\right)^{5}$

$(5,4,3,2,1)$ $\quad$ $(5,4,3,2,1)$

(iv) $P(X<4) = P(X \leq 3) = 1 - q^{r} = 1 - \left(\frac{5}{6}\right)^{3}$

$(3,2,1)$ $\quad$ $(3,2,1)$

(v) $P(X>7) = q^{r} = \left(\frac{5}{6}\right)^{7}$



(vi) $P(X \geq 4) = P(X>3) = q^{3} = \left(\frac{5}{6}\right)^{3}$

$$\boxed{\begin{aligned} P(X \leq r) &= 1 - q^{r} \\ P(X > r) &= q^{r} \end{aligned}}$$

1 The score when two fair six-sided dice are thrown is the sum of the two numbers on the upper faces.

(a) Show that the probability that the score is 4 is $\frac{1}{12}$. (AG) [1]

Total = 36

Sum is 4 : $P(\text{Sum} = 4) = \frac{3}{36} = \frac{1}{12}$

(1,3)

(3,1)

(2,2)

The two dice are thrown repeatedly until a score of 4 is obtained. The number of throws taken is denoted by the random variable X.

Geometric

$p = \frac{1}{12}$, $q = \frac{11}{12}$ from last part

(b) Find the mean of X. [1]

$\text{Mean} = \frac{1}{p} = \frac{1}{\frac{1}{12}}$

Mean = 12

(c) Find the probability that a score of 4 is first obtained on the 6th throw. [1]

$P(X=6) = q^5 p = \left(\frac{11}{12}\right)^5 \left(\frac{1}{12}\right) = 0.05394$

(d) Find $P(X < 8)$. [2]

$P(X \leq 7) = 1 - q^r$ $\qquad P(X \leq r) = 1 - q^r$

$= 1 - \left(\frac{11}{12}\right)^7$

$= 0.456$

7 On any given day, the probability that Moena messages her friend Pasha is 0.72.

(a) Find the probability that for a random sample of 12 days Moena messages Pasha on no more than 9 days. [3]

Binomial : $n=12$, $p=0.72$, $q=0.28$

$P(X \leq 9) = 1 - P(10, 11, 12)$

$= 1 - \left[{}^{12}C_{10}(0.72)^{10}(0.28)^2 + {}^{12}C_{11}(0.72)^{11}(0.28)^1 \; {}^{12}C_{12}(0.72)^{12}(0.28)^0 \right]$

$= 1 - (0.19372 + 0.09057 + 0.01941)$

$= 0.696$

n

$P(x<9)$ $\boxed{\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ 9 \end{matrix}}$

10

11

12

(b) Moena messages Pasha on 1 January. Find the probability that the next day on which she messages Pasha is 5 January. [1]

2 Jan	3 Jan	4 Jan	5 Jan
F	F	F	S

F^3S

$(0.28)^3(0.72) = 0.01581$

5 A pair of fair coins is thrown repeatedly until a pair of tails is obtained. The random variable X denotes the number of throws required to obtain a pair of tails.

geometric

HH, HT, TH, TT

(a) Find the expected value of X. [1]

$p = \frac{1}{4}, q = \frac{3}{4}$ $P(TT) = \frac{1}{4}$

Mean = Expected value $= \frac{1}{p} = \frac{1}{\frac{1}{4}} = \boxed{4}$

(b) Find the probability that exactly 3 throws are required to obtain a pair of tails. [1]

$P(X=3) = q^2 p = \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) = 0.141$

(c) Find the probability that fewer than 6 throws are required to obtain a pair of tails. [2]

$P(X<6)$

$P(X \leq r) = 1 - q^r$

$P(X \leq 5) = 1 - q^r = 1 - \left(\frac{3}{4}\right)^5$

$= 0.7627$

3 Kayla is competing in a throwing event. A throw is counted as a success if the distance achieved is greater than 30 metres. The probability that Kayla will achieve a success on any throw is 0.25.

(a) Find the probability that Kayla takes more than 6 throws to achieve a success. [2]

Geometric: $p = 0.25$, $q = 0.75$

$P(X>6) = q^r = 0.75^6$ $\quad P(X>r) = q^r$

$= 0.178$

(b) Find the probability that, for a random sample of 10 throws, Kayla achieves at least 3 successes. [3]

Binomial: $n = 10$, $p = 0.25$, $q = 0.75$

$P(X \geq 3) = 1 - P(0, 1, 2)$

$\boxed{X}$

0

1

2

$P(X \geq 3)$ at least three success

3

4

5

6

⋮

9

10

1 A fair six-sided die, with faces marked 1, 2, 3, 4, 5, 6, is thrown repeatedly until a 4 is obtained.

(a) Find the probability that obtaining a 4 requires fewer than 6 throws. geometric [2]

$p = \frac{1}{6}$, $q = \frac{5}{6}$

Fewer than 6

$P(X < 6)$ $\quad P(X \leq r) = 1 - q^r$

$P(X \leq 5) = 1 - q^r$

$= 1 - \left(\frac{5}{6}\right)^5 = 0.598$

On another occasion, the die is thrown 10 times.

(b) Find the probability that a 4 is obtained at least 3 times. [3]

Binomial $\quad n = 10$, $p = \frac{1}{6}$, $q = \frac{5}{6}$

$P(X \geq 3) = 1 - P(0, 1, 2)$

2 An ordinary fair die is thrown until a 6 is obtained. Geometric. $p=\frac{1}{6}, q=\frac{5}{6}$

(a) Find the probability that obtaining a 6 takes more than 8 throws. [2]

$P(X>8) = q^r = \left(\frac{5}{6}\right)^8$ $\quad P(X>r)=q^r$

$= 0.233$

Two ordinary fair dice are thrown together until a pair of 6s is obtained. The number of throws taken is denoted by the random variable X.

geometric $\Rightarrow p=\frac{1}{36}, q=\frac{35}{36}$

(b) Find the expected value of X. [1]

$\text{Mean} = \frac{1}{p} = \frac{1}{\frac{1}{36}} = 36$

(c) Find the probability that obtaining a pair of 6s takes either 10 or 11 throws. [2]

$P(X=10) + P(X=11)$

$q^9 p + q^{10} p$

$\left(\frac{35}{36}\right)^9\left(\frac{1}{36}\right) + \left(\frac{35}{36}\right)^{10}\left(\frac{1}{36}\right) = 0.0425.$